Theoretical model for the pulse dynamics in a long granular chain

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(Received 29 May 2006; published 3 October 2006)

When a long line of initially stationary and unstressed touching particles is hit on one end by another particle, dynamical self-organized impulsive waves will be generated and propagating through the granular chain. In this paper we develop a continuous function to represent the initial impulse wave. Then the partial differential wave equations for chains of cylindrical granules, which can take such function as one of its solutions, are constructed and exactly coincided with the wave equation obtained by using Taylor series expansion. The properties of the impulse wave propagating, such as the attenuation of the wave and the forward momentum transfer are studied in detail based on our theoretical model, and the analytical solutions are supported by numerical simulations.

DOI: 10.1103/PhysRevE.74.046602 PACS number(s): 46.40.−f, 45.05.+x, 45.70.−n

I. INTRODUCTION

The propagation of wave in granular chains with particles interacting according to the contact law has triggered the vigorous interest of many researchers [1–25]. The important technological application of this new area can be found in the design of efficient shock absorber [17,21], the detection of buried objects [3], and the creation of nanodroplets [5].

The shape of the granules, the details of the contact law between adjacent particles, and the static compressive force all affect the propagation of the impulse wave in granular material [2]. Due to the complexity of such problem, numerical solution of the equations of motion established by adding compliance is often used to investigate the nonlinear behavior of a granular chain. However, searching for an effective continuum approximation and finding an analytical solution to the discrete system can considerably help us to understand the global qualities of the granular chain, particularly to study the propagation of shock waves.

Nesterenko [1] showed that under appropriate assumptions the equations of motion for granular particles could be approximated by a continuous nonlinear partial differential equation and provide an analytical solution to the problem for Hertz's contact law. Such solution represents an isolated solitary wave that propagates the perturbation in the chain. His theory has been verified by many researchers and found a good agreement with the properties of these waves investigated by using numerical and experimental methods [2,22–24].

Although a purely isolated solitary wave cannot be generated in a cylindrical granular chain, which the interaction between particles is represented by a linear contact force, a similar behavior of a shock wave propagating in the chain was also discovered in Refs. [4,13]. Hinch and Saint-Jean (HSJ) obtained the wave equation of continuum approximation by using Taylor series expansion and then sought a similarity solution to the wave equation. Rosas [13] comprehensively summarized HSJ theory and developed some analytical results for the cylindrical granular chain on the basis of HSJ theory. Since many conclusions in HSJ theory are based on their numerical observations and qualitative analysis, it still needs to justify such theory. For example, whether the wave equation in terms of Taylor series expansion is accurate enough to describe the continuous properties of the granular chain, what the function of the impulse wave is that it can absolutely satisfy the wave equation, how the extent of attenuation of the impulse wave is when it is traveling in the chain.

In this paper, we make an attempt to carry out such work. When the chain is hit by another particle, an initial shock wave will be generated due to the self-organized behavior between particles, and its shape will not be influenced by the length of the chain. In other words, a correct shape of the initial impulse wave can be formed if the number of particles in the chain is greater than a minimum. Thus, it is possible to obtain an explicit function to represent the shape of the initial wave by using theoretical method to analyze a shorter chain. Once such function of the initial wave profile is found, a partial differential equation for describing the continuous properties of the granular chain, which should take such function as one of its solutions, might be constructed by using mathematical tools. Furthermore, under some appropriate assumptions for a long chain, the properties of the impulse wave propagating can be studied by using the theoretical model.

Based on the idea mentioned above, we present the study as follows. First we establish a general theoretical method for a shorter chain to get the analytical solutions to the motion of each particle. The analytical process is illustrated by a chain with 10 particles. Second we study the influencing scope of the initial impact for the chains with different lengths. We find that the correct shape of an initial wave can be formed in a seven particles chain, and is not further affected by increase of the chain's length. Third we use the theoretical results of the chain with seven particles to find the function of the initial impulse wave, and construct the partial differential equation of continuum approximation to the cylindrical granular chains. We find that such partial differential equation coincides well with the results obtained by using

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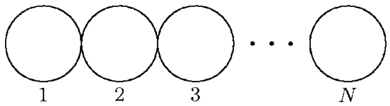

FIG. 1. Theoretical model of chain with N particles.

Taylor series expansion. Finally, by introducing the assumptions that the profile of the impulse wave is constant, we probe the properties of the wave propagating, and obtain comprehensively analytical results for the chains of cylindrical granules. Summary and conclusions are presented at the end of this paper.

II. DESCRIPTION OF 1D GRANULAR CHAIN

One-dimensional (1D) granular chain is keeping touch, but initially stationary and unstressed, as shown in Fig. 1. The centroid of each ball is located in a line to ensure only axial impact happens after the chain sustains a shock.

Suppose the granular chain consists of N particles with the mass m having the contact force $F=k\delta^{\alpha}$, where k is an elastic constant coefficient, δ is the penetration between two particles, and the exponent α determine the kind of contacts between the particles. For instance, $\alpha=1$ for solid cylinders or spherical shells, $\alpha=\frac{3}{2}$ for the Hertzian interaction between solid spherical balls, whereas $\alpha=2$ defines an interaction with conical asperities. This paper only considers the case of $\alpha=1$.

Denote u_i as the displacement of the particle i from its initial touching positions. The equations of motion of the particle number i is the following:

$$m\frac{d^2u_i}{dt^2}=k(u_{i-1}-u_i)_+-k(u_i-u_{i+1})_+ \tag{1}$$

The notation + indicates that the contact force is zero when the particles lose contact, which means the contact force cannot be in tension. For $i=1$, the first term on the right-hand side of Eq. (1) is dropped; while for $i=N$ the second term is dropped.

If the first particle $(i=1)$ is the impacting particle with speed v_0, the initial conditions for the system are

$$u_1(0)=u_2(0)=\cdots=u_N(0)=0,$$

$$\dot{u}_1(0)=v_0,\dot{u}_2(0)=\cdots=\dot{u}_N(0)=0. \tag{2}$$

Before the impacting particle rebounds backwards, all the particles in the chain keep contact. Therefore, we consider the granular chain as a continuous system. The equations of motion of the chain can be expressed in a linear space,

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{u}\\ \dot{\boldsymbol{u}}\end{bmatrix}=\boldsymbol{H}\begin{bmatrix}\boldsymbol{u}\\ \dot{\boldsymbol{u}}\end{bmatrix}, \tag{3}$$

where

$$\boldsymbol{H}=\left[\begin{array}{cc} 0_{N\times N} & I_{N\times N} \\ \begin{array}{cccc} -\omega^2 & \omega^2 & \cdots & 0 \\ \omega^2 & -2\omega^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \omega^2 & -\omega^2 \end{array} & 0_{N\times N} \end{array}\right],$$

$$\boldsymbol{u}=[u_1,u_2,\cdots,u_N]^T,\ \dot{\boldsymbol{u}}=[\dot{u}_1,\dot{u}_2,\cdots,\dot{u}_N]^T,$$

and $\omega=\sqrt{k/m}$.

Define the following dimensionless variables space:

$$\boldsymbol{v}(t)=[\boldsymbol{\zeta},\boldsymbol{\eta}]^T,$$

where $\boldsymbol{\zeta}=\boldsymbol{u}/(v_0\cdot 1s)$, $\boldsymbol{\eta}=\dot{\boldsymbol{u}}/v_0$. The equation of motion for the continuous chain can be expressed as

$$\frac{d}{dt}\boldsymbol{v}=\boldsymbol{H}\boldsymbol{v} \tag{4}$$

with initial condition $\boldsymbol{v}_0=[0,0,\cdots,0,1,0,\cdots,0]^T$.

Theoretical solution of Eq. (4) is

$$\boldsymbol{v}(t)=\exp(\boldsymbol{H}t)\cdot\boldsymbol{v}_0. \tag{5}$$

From matrix algebraic theory, we can decompose matrix $\boldsymbol{H}$ in the following form:

$$\boldsymbol{H}=\boldsymbol{P}^{-1}\boldsymbol{J}\boldsymbol{P}, \tag{6}$$

where $\boldsymbol{J}$ is the Jordan canonical form of $\boldsymbol{H}$,

$$\boldsymbol{J}=\begin{bmatrix} \begin{matrix}0 & 1\\ 0 & 0\end{matrix} & & & & 0 \\ & \lambda_1\omega i & & & \\ & & -\lambda_1\omega i & & \\ 0 & & & \ddots & \\ & & & & -\lambda_{N-1}\omega i \end{bmatrix}.$$

$\boldsymbol{J}$ includes two zero roots, and $N-1$ pairs of conjugate pure imaginary roots. The two zero roots represent the common rigid movement of the chain, and the $N-1$ pairs of conjugate pure imaginary roots reflect the interaction between particles.

Matrix function $\exp(\boldsymbol{H}t)$ in Eq. (5) can be written by

$$\exp(\boldsymbol{H}t)=\boldsymbol{P}^{-1}\exp(\boldsymbol{J}t)\boldsymbol{P}. \tag{7}$$

Indeed, the theoretical solution expressed in Eq. (5) comes into existence if the chain does not break. After the first particle loses contact with the chain, we can treat other particles which still keeping contact as a new shorter chain, and take the states of the previous chain at the moment that detachment occurs as the initial conditions of the new chain. Repeating the procedure mentioned above we can obtain the theoretical solutions of the new chains. Obviously, the final state of each particle after impact can be obtained in this method, and the theoretical results of the chain have the form of piecewise function.

TABLE I. The eigenvalues of matrix $\boldsymbol{J}$ for a chain with 10 particles.

λ_1	0.3129	λ_2	0.6180	λ_3	0.9080
λ_4	1.1756	λ_5	1.4142	λ_6	1.6180
λ_7	1.7820	λ_8	1.9021	λ_9	1.9754

III. THEORETICAL RESULTS FOR A CHAIN WITH 10 PARTICLES

To illustrating the theoretical process, we consider a short chain with 10 particles which have the same mass and stiffness. The matrix $\boldsymbol{H}$ for such chain can be written as

$$\boldsymbol{H} = \begin{bmatrix} & 0_{10\times 10} & & & I_{10\times 10} \\ -\omega^2 & \omega^2 & \cdots & 0 & \\ \omega^2 & -2\omega^2 & \cdots & 0 & 0_{10\times 10} \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & -\omega^2 & \end{bmatrix}$$

The eigenvalues λ_i in the matrix $\boldsymbol{J}$ for such chain are shown in Table I.

By using MAPLE tools, we can get the displacement and the velocity of each particle before the chain is broken,

$$\zeta_i = \frac{1}{10}t + \sum_{j=1}^{9} \frac{b_{ij}}{\omega}\sin(\lambda_j \omega t) \quad (i = 1,2,\cdots,10), \tag{8a}$$

$$\eta_i = \frac{1}{10} + \sum_{j=1}^{9} b_{ij}\lambda_j \cos(\lambda_j \omega t) \quad (i = 1,2,\cdots,10). \tag{8b}$$

The coefficients b_{ij} in the above equations are not shown in this paper. The displacement of the first particle equals the second at the moment t^*, after that the first particle will detach from the chain since the contact force cannot be in tension. In terms of $\zeta_1(t^*)=\zeta_2(t^*)$ we can obtain $t^*=2.5681/\omega$. After the first particle rebounds with velocity $\eta_1(t^*)=-0.1323$, other particles still keep contact and form a new chain with the initial condition at the instant t^*.

Repeat of the analysis step can obtain the theoretical results for all the particles. The displacement of the first particle can be expressed as a piecewise function with two parts,

$$\zeta_1(t) = \begin{cases} 0.1t + \displaystyle\sum_{j=1}^{9} \frac{b_{1j}}{\omega}\sin(\lambda_j \omega t) & (t \le 2.5681/\omega), \\ \dfrac{1.3737}{\omega} - 0.1323t & (t > 2.5681/\omega). \end{cases} \tag{9}$$

The displacement of the second particle can be expressed as a piecewise function with three parts [define $\tau=\omega t - 2.5681$ in Eq. (10)],

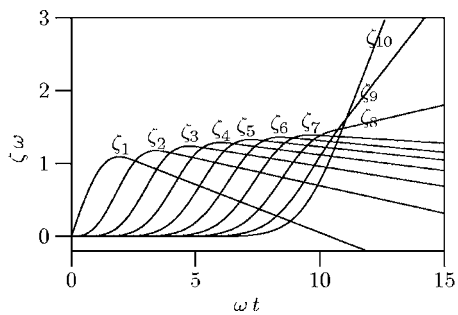

FIG. 2. The displacements of each particle for a chain with 10 particles.

$$\zeta_2(t) = \begin{cases} 0.10t + \sum_{j=1}^{9} \dfrac{b_{2j}}{\omega}\sin(\lambda_j \omega t) \quad (t \le 2.5681/\omega), \\ 0.8878t + \dfrac{0.6129}{\omega}\sin(0.3129\tau), \\ + \dfrac{0.2937}{\omega}\sin(0.6180\tau) + \dfrac{0.1659}{\omega}\sin(0.9080\tau), \\ + \dfrac{0.1094}{\omega}\sin(1.1756\tau) + \dfrac{0.0679}{\omega}\sin(1.4142\tau), \\ + \dfrac{0.0386}{\omega}\sin(1.6180\tau) + \dfrac{0.0199}{\omega}\sin(1.7820\tau), \\ + \dfrac{0.0087}{\omega}\sin(1.9021\tau) + \dfrac{0.0019}{\omega}\sin(1.9754\tau), \\ (1.3231 \ge \tau > 0), \\ \dfrac{0.2643}{\omega} - 0.0754t \quad (t > 3.8912/\omega). \end{cases} \tag{10}$$

The second particle will lose contact with the chain at the moment $2.5681/\omega$. Analogically, we can express the displacement of the third particle as a piecewise function with four parts, and the fourth particle with five parts, etc. Figures 2 and 3 show the displacement and the velocity of all the particles in the chain, respectively. From the figure we can find that the displacements become linear in time, which means all the contacts are lost eventually; and the velocities are constant after the particles detach from the chain. The final speed of each particle is −0.1323, −0.0754, −0.0547, −0.0436, −0.0366, −0.0316, −0.0213, 0.0788, 0.4389, and

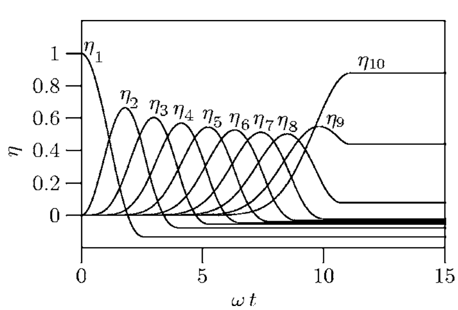

FIG. 3. The velocity of each particle for a chain with 10 particles.

TABLE II. The eigenvalues λ^7 for a chain with seven particles.

λ_1^7	0.4450	λ_2^7	0.8678	λ_3^7	1.2470
λ_4^7	1.5637	λ_5^7	1.8019	λ_6^7	1.9499

0.8777. Obviously, seven out of 10 particles rebound backwards, and about 96.91% of impacting kinetic energy will be taken by the last three particles moving forward.

The parameter ω cannot influence the final speed of all the particles, which means the kinetic energy and momentum that transfer forward are not affected by the material properties of the particle. But the material properties will influence the transferring time with proportion to inversion with ω, in other words, the greater the contact stiffness is, the shorter the transferring time.

IV. THE INITIAL LENGTH OF THE IMPULSE WAVE

In Fig. 3 we show that a impulse wave will be generated and propagates down the chain. The particles are at rest ahead of the wave, and it will rebound backward with negative velocity after the wave has passed. From the theoretical solutions of the 10 particles chain, we find that the velocity of the last particle equals zero at $t^*=2.5681/\omega$, and t^* is the time that the first particle detaches from the chain. This implies the initial impulse wave has no influence upon the last particle before the first particle loses contact. To determine the initial length of the impulse wave, we present the theoretical solutions for a granular chain with only seven particles. The displacement for each particle of seven particles chain can be expressed as

$$\zeta_i^7 = \frac{1}{7}t + \sum_{j=1}^{6} \frac{b_{ij}^7}{\omega} \sin(\lambda_j^7 \omega t) \quad (i = 1,2, \ldots ,7). \qquad (11)$$

The coefficients, λ_i^7 and b_{ij}^7, are shown in Tables II and III.

Based on the theoretical solutions for the seven particles chain, we find the time that the first particle detaches from the chain is $t^*=2.5681/\omega$, it is exactly equal to the one of the 10 particles chain. Meanwhile, the velocity of the last particle for the seven chain at t^* is almost zero. That means the initial disturbance cannot influence the seventh particle before the first particle rebounds backwards.

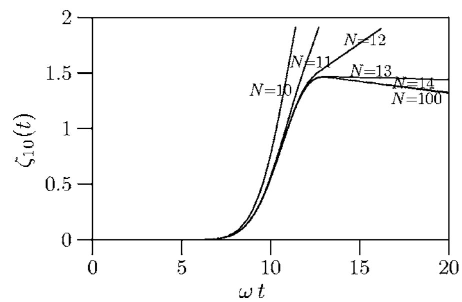

FIG. 4. The displacement of the tenth particle in the chain with different number of particles. If $N=10$, that also means the tenth particle is the last particle of the chain, after impact the velocity of the tenth particle will get positive maximum. With the increasing of the chain length N, the positive velocity of such particle will decrease and finally become negative. The absolute value of the rebounding velocity will increase with the length of the chain and get a limit value if the chain is long enough. After the rebounding velocity of the tenth ball reaches the limit value, the length of the chain N cannot affect the movement of the tenth ball anymore.

Similarly, the theoretical solutions for the chain with six particles can be expressed as

$$\zeta_i^6 = \frac{1}{6}t + \sum_{j=1}^{5} \frac{b_{ij}^6}{\omega} \sin(\lambda_j^6 \omega t) \quad (i = 1,2, \ldots ,6). \qquad (12)$$

In this case, the time that the first particle detaches from the chain equals $2.5678/\omega$. It is shorter than the chain with seven or 10 particles. The velocity of the last particle for the chain with six particles is $\eta_6(t^*)=0.002$, that means a whole shape of impulse wave cannot be generated in the chain with six particles. Thus, we can confirm that the initial length of the impulse wave only involves five particles, i.e., from the second particle to the sixth particle.

Figure 4 shows the displacement of the tenth particle in the chains with a different number of particles. The impulse wave will propagate down the chain, the tenth particle will get positive velocity if the distance between the tenth particle and the end of the chain is smaller than half-wavelength, and it reaches minimum of positive velocity when such distance equals the half-wavelength. If such a distance is greater than

TABLE III. The coefficient b_{ij}^7 for a chain with seven particles.

i \ j	1	2	3	4	5	6
1	0.6103	0.2673	0.1400	0.0710	0.0299	0.0073
2	0.4892	0.0660	−0.0777	−0.1027	−0.0671	−0.0203
3	0.2717	−0.1850	−0.1746	−0.0254	0.0538	0.0294
4	0.0000	−0.2966	0.0000	0.1139	0.0000	−0.0326
5	−0.2716	−0.1849	0.1746	−0.0254	−0.0538	0.0294
6	−0.4893	0.0660	0.0777	−0.1027	0.0671	−0.0203
7	−0.6102	0.2673	−0.1401	0.0710	−0.0299	0.0073

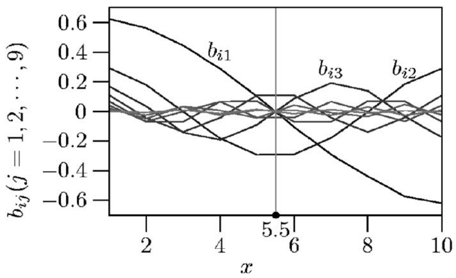

FIG. 5. The coefficients b_{ij} in a chain with 10 particles. Each b_{ij} is located at a different line b_{i1}, $b_{i2}, \ldots,$ and b_{i9}, respectively.

the half-wavelength, the tenth particle will lose its contact with rebounding motion and its rebounding velocity will increase with the length of the chain. When the distance between the tenth particle and the end of the chain is greater than a complete wavelength, the limit value of the rebounding velocity will be reached and the motion of the tenth particle cannot be influenced by the further increase of the length of the chain.

V. THE FUNCTION OF THE INITIAL IMPULSE WAVE AND THE WAVE EQUATION

If the granular chain is long enough, the initial impulse wave will be generated before the first particle loses contact. In the following we will determine the initial shape of the impulse wave in terms of the theoretical solutions.

Observing the theoretical solutions in Eqs. (8a), (11), and (12), we can write the continuous displacement function $u(x,t)$ for a chain with N particles as the following form:

$$u(x,t) = \frac{1}{N}t + \sum_{j=1}^{N-1} \frac{f_j(x)}{\omega} \sin(\lambda_j \omega t), \tag{13}$$

where $f_j(i)=b_{ij}$ (and $i,j=1,\cdots,N$).

Obviously we have $\zeta_i=u(i,t)$. Observing the changes of the coefficients, b_{ij} in Table II (Fig. 5 shows b_{ij} for a chain with 10 particles), the continuous function $f_j(x)$ can be written by

$$f_j(x) = A_j \sin\left[\frac{j\pi}{N}\left(x - \frac{N+1}{2}\right) + (-1)^j \frac{j+1}{2}\pi\right]. \tag{14}$$

The undetermined coefficients, A_j can be determined by using least squares method.

Define $\Delta_j=\sum_{i=1}^{N}[f_j(i)-b_{ij}]^2$, where b_{ij} is the distance between the particle i and j, we have

$$\frac{\partial \Delta_j}{\partial A_j} = 0 \quad (j=1,\ldots,N-1).$$

Then the coefficients A_j can be obtained as follows:

$$A_j = \frac{\sum_{i=1}^{N} b_{ij} \sin\left[\frac{j\pi}{N}\left(i - \frac{N+1}{2}\right) + (-1)^j \frac{j+1}{2}\pi\right]}{\sum_{i=1}^{N} \sin^2\left[\frac{j\pi}{N}\left(i - \frac{N+1}{2}\right) + (-1)^j \frac{j+1}{2}\pi\right]}.$$

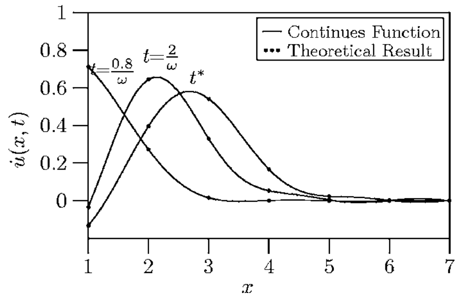

FIG. 6. The real line is the continuous function $\dot{u}(x,t)$ from Eq. (16) at different time $0.8/\omega$, $2/\omega$, and $t^*=2.5681/\omega$. The discrete points on each line are the theoretical solutions of each particle for seven particles chain at the same moment.

Since the initial disturbance only involves five particles and the initial impulse wave has been formed before the first particle rebounds backwards, we can use the theoretical solutions for the chain with seven particles to determine the continuous functions $f_j(x)$. In terms of the coefficients b_{ij} in Table III, $f_j(x)$ have the following form:

$$f_1(x) = 0.6259 \sin\left(\frac{\pi}{7}x - \frac{11\pi}{7}\right), \tag{15a}$$

$$f_2(x) = 0.2967 \sin\left(\frac{2\pi}{7}x + \frac{5\pi}{14}\right), \tag{15b}$$

$$f_3(x) = 0.1791 \sin\left(\frac{3\pi}{7}x - \frac{26\pi}{7}\right), \tag{15c}$$

$$f_4(x) = 0.1139 \sin\left(\frac{4\pi}{7}x + \frac{3\pi}{14}\right), \tag{15d}$$

$$f_5(x) = 0.0688 \sin\left(\frac{5\pi}{7}x - \frac{41\pi}{7}\right), \tag{15e}$$

$$f_6(x) = 0.0326 \sin\left(\frac{6\pi}{7}x + \frac{\pi}{14}\right). \tag{15f}$$

The continuous function for describing the displacement of the initial impulse wave can be expressed as

$$u(x,t) = \frac{1}{7}t + \sum_{j=1}^{6} \frac{f_j(x)}{\omega} \sin(\lambda_j \omega t). \tag{16}$$

Of course we can use the theoretical results of the chain with more particles to establish the continuous function. It has little influence on the accuracy of the continuous function.

So far, we have obtained a wave-type continuous function $u(x,t)$. Figure 6 shows the velocity of the impulse wave as a continuous function $\dot{u}(x,t)$ compared with the theoretical solutions for a chain with seven particles at different time. An initial impulse wave has been generated at the time $t=2/\omega$ due to the self-organized interactions between particles, and the continuous function $\dot{u}(x,t)$ can well reflect the propagating of the wave before the first particle detaches from the chain at time $t^*=2.5681/\omega$.

If the chain is not broken, we can take the granular chain as a continuum body. The continuous function $u(x,t)$ can be considered as a solution of the wave equation of the continuum approximation for the granular chain. Since $u(x,t)$ is a combination of a series of trigonometric functions and with a form of separating space and time, the terms of the partial derivative of $u(x,t)$ on x with odd order cannot appear in the partial wave equation. Thus, the wave equation should have the form as follows:

$$\frac{\partial^2 u}{\partial t^2} = \omega^2\left(P_1\frac{\partial^2 u}{\partial x^2} + P_2\frac{\partial^4 u}{\partial x^4} + P_3\frac{\partial^6 u}{\partial x^6} + \cdots\right), \qquad (17)$$

where P_i are the coefficients that need to be decided. Taking continuous function (13) into (17) can have

$$\left(\frac{j\pi}{N}\right)^2 P_1 + \cdots + (-1)^N\left(\frac{j\pi}{N}\right)^{2(N-1)} P_{N-1} = \lambda_j^2. \qquad (18)$$

For a continuous function $u(x,t)$ obtained from the theoretical solutions of a chain with N particles, there are $(N-1)$ independent algebraic equations with the form of Eq. (18). Then, the undetermined coefficients P_i in Eq. (17) can be calculated.

Since an impulse wave can be formed in a chain with seven particles, we can take the parameters λ_j in Table II into Eq. (18) to determine P_i. In this case, $P_1=1.0000$, $P_2=8.3333\times10^{-2}$, $P_3=2.7778\times10^{-3}$, $P_4=4.9576\times10^{-5}$, $P_5=5.3259\times10^{-7}$, $P_6=-2.4480\times10^{-9}$.

Except the first two terms P_1 and P_2, other terms have a very little contribution for the wave equation (17). Meanwhile, even if we take the theoretical solutions to the chain with more particles than seven, the changes of the coefficient P_i are very little. We can get a simplified wave equation as follows:

$$\frac{\partial^2 u}{\partial t^2} = \omega^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}\right). \qquad (19)$$

It is very interesting that Eq. (19) is well agreement with the result of HSJ theory which based on the Taylor series expansion and only retaining the first leading term. This illustrates the HSJ theory is sufficiently accurate. Continuous function $u(x,t)$ expressed in Eq. (16) can be taken as the wave-type solution for a granular chain before it is broken.

VI. THE ATTENUATION OF THE IMPULSE WAVE PROPAGATING

Before the first particle loses contact, from Fig. 6 we can find that the initial profile of speed wave can well represented by the function $\dot{u}(x,t)$. Taking the time t^*, the instant that the first particle rebounds backwards, as a referenced point, we can obtain a correct profile of the impulse wave in terms of Eq. (16),

$$g(x) = \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=t^*} = \frac{1}{7} + s(x), \qquad (20)$$

where

$$s(x) = 0.1156\sin\left(\frac{\pi}{7}x + \frac{3\pi}{7}\right) - 0.1574\sin\left(\frac{2\pi}{7}x + \frac{5\pi}{14}\right) - 0.2229\sin\left(\frac{3\pi}{7}x + \frac{2\pi}{7}\right) - 0.1143\sin\left(\frac{4\pi}{7}x + \frac{3\pi}{14}\right) - 0.0105\sin\left(\frac{5\pi}{7}x + \frac{\pi}{7}\right) + 0.0185\sin\left(\frac{6\pi}{7}x + \frac{\pi}{14}\right).$$

At this moment, the wave reaches the position $x_f=6.0622$, where the velocity equals zero $g(x_f)=0$; and the tail of the wave is located at $x_b=1.2928$, where $g(x_b)=0$. The crest of the wave stands at the position $x_{\max}=2.6835$ with maximum $g(x_{\max})=0.5818$. The impulse wave is strongly localized with most of the kinetic energy in just five particles at this time.

Because the discrete effects of the rebounding motion of the particles, the initial impulse wave will move as a spreading wave and shed some energy. This will make the wavelength increases and its hump decreases. To analyze the attenuation of the wave propagating, we follows the assumption presented in HSJ theory.

The impulse wave propagates through the chain with a constant profile at constant speed ω, and it has an amplitude and width that vary slowly with time. Then we can assume the function of the impulse wave has the following form:

$$V(x,t) = a(t)g\left(\frac{x - x_{\max} - \omega t}{\lambda(t)} + x_{\max} + \omega t^*\right) = a(t)g\left(\frac{x - x_{\max} - \omega t}{\lambda(t)} + 5.2515\right) \qquad (21)$$

and

$$x_b \le \frac{x - x_{\max} - \omega t}{\lambda(t)} + 5.2515 \le x_f,$$

where $a(t)$ and $\lambda(t)$ represent the functions that the amplitude and width of the impulse wave change with time. We substitute $\left(\frac{x-x_{\max}-\omega t}{\lambda(t)} + x_{\max} + \omega t^*\right)$ for the variable x in the function $g(x)$. Obviously we have $V(x,t^*)=g(x)$, $a(t^*)=1$, and $\lambda(t^*)=1$, which represents the shape of the impulse wave at time t^*.

Then, the kinetic energy involved in the impulse wave at time t can be written by

$$E = \sum \frac{1}{2}V^2(n,t) = \frac{1}{2}a^2(t)\sum g_n^2 = \frac{1}{2}a^2(t)\int g^2\left(\frac{x - x_{\max} - \omega t}{\lambda(t)} + 5.2515\right)dx. \qquad (22)$$

Introducing a coordinate transformation variable

$$y = \frac{x - x_{\max} - \omega t}{\lambda(t)} + 5.2515$$

$x_b \le y \le x_f$ from Eq. (22). We can get

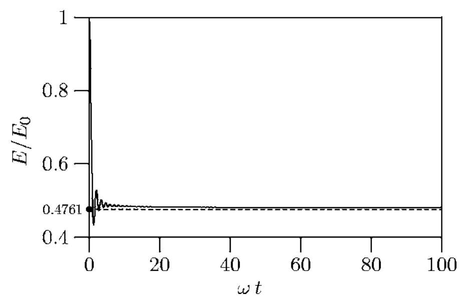

FIG. 7. Kinetic energy of the particles moving forward in a chain with 500 particles. Initially, the kinetic energy moving forward will be shakable due to the scattering motion of the particles with negative rebounding velocity. It is clearly observed that such energy will be rapidly stable and approach an invariable value.

$$\int g^2\left(\frac{x - x_{\max} - \omega t}{\lambda(t)} + 5.2515\right)dx = \int_{x_b}^{x_f} g^2(y)d[\lambda(t)y] = 0.4761\lambda(t). \qquad (23)$$

Then the kinetic energy of the impulse wave can be expressed as

$$E = \tfrac{1}{2}0.4761a^2(t)\lambda(t). \qquad (24)$$

In the HSJ theory, the pulse is assumed to retain almost all of its initial kinetic energy based on numerical observation. We can also find from the theoretical solutions for the chain with 10 particles that the rebounding velocity of the detaching particles decreases rapidly and approaches zero. In fact, with the increasing of the chain's length, the particles that detaches from the chain will have zero rebounding velocity. This implies that the kinetic energy involved in the wave approaches a constant.

Figure 7 shows the changes of the kinetic energy involved in the wave with time based on our numerical results for a chain with 500 particles. It is interesting that the ratio of such kinetic energy to the initial kinetic energy of the impacting particle ($E_0 = 1/2$) rapidly approaches to 0.4761. In terms of Eq. (24), we have

$$E = \tfrac{1}{2}0.4761a^2(t)\lambda(t) = \tfrac{1}{2}0.4761. \qquad (25)$$

Then, we can get an important relationship between $a(t)$ and $\lambda(t)$,

$$a^2(t)\lambda(t) = 1. \qquad (26)$$

Now, the wave function $V(x,t)$ can be expressed as

$$V(x,t) = a(t)g[a^2(t)(x - x_{\max} - \omega t) + 5.2515]. \qquad (27)$$

This wave function also automatically satisfies the partial differential equation of the continuum approximation to the granular chain expressed in Eq. (19). Similar to the method in HSJ theory, we make a coordinate transformation

$$v = x - x_{\max} - \omega t.$$

The equation (19) becomes

$$\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^2 u}{\partial t\, \partial v} = \frac{1}{12}\frac{\partial^4 u}{\partial v^4}, \qquad (28)$$

$u(v,t)$ can be written by

$$u(v,t) = \int_0^t V(x,t)dt = \int_0^t a(t)g[a^2(t)v + 5.2515]dt. \qquad (29)$$

Taking (29) into (28) leads to

$$a'g + 2a^2vg^{(1)}a' - 2a^3g = \frac{1}{12}\int_0^t a^9g^{(4)}dt. \qquad (30)$$

The notation $(\cdot)'$ represents the derivative of time, and the notation $(\cdot)^{(\cdot)}$ represents the partial derivative of the composite function $g(v,t)$. Since $a(t)$ is a variable varying slowly with time, we can omit the terms involving a' on the right-hand side of Eq. (30). Then

$$-24a^3g \approx \int_0^t a^9g^{(4)}dt. \qquad (31)$$

Differentiating Eq. (31) with t,

$$-24(3g + 2a^2g^{(1)}v)a' \approx a^7g^{(4)}. \qquad (32)$$

The second term on the left-hand side of Eq. (32) has smaller value with higher order compared with the first term. Then, we can get an approximate relationship

$$-72ga' \approx a^7g^{(4)}.$$

The function of the wave profile is the sum of a series of trigonometric functions, making the above equation be compatible, $a(t)$ should have the following form:

$$a' \propto -a^7, \quad a(t) \propto t^{-1/6}.$$

Notice that $a(t^*)=1$, $\lambda(t^*)=1$ at $t^*=2.5681/\omega$, and $a^2(t)\lambda(t)=1$, then

$$a(t) = 1.1702(\omega t)^{-1/6}, \quad \lambda(t) = 0.7302(\omega t)^{1/3}. \qquad (33)$$

From Eqs. (21) and (33), we have obtained the function of the impulse wave propagating along a long granular chain. To check the accuracy of the analytical results, we use numerical simulations to solve the dynamical equations for a chain with 1000 particles. We follow HSJ and plot the scaled velocity as a function of the scaled position for different times in Fig. 8 and we can find that the numerical results agree well with our analytical form of the impulse wave.

The time that the wave reaches the particle n is

$$\frac{n - x_{\max} - \omega t}{\lambda(t)} + 5.2515 = x_f.$$

Then, we have a curve that represents the position of the front of the wave at time t,

$$n = 2.6835 + \omega t + 0.5920(\omega t)^{1/3}.$$

Similarly, the curve that represents the position of the rear of the wave at time t is

$$n = 2.6835 + \omega t - 2.8906(\omega t)^{1/3}.$$

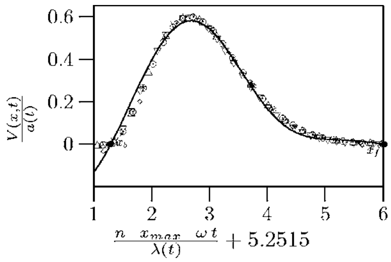

FIG. 8. The real line is the initial profile of the wave at t^*. The symbols (◇, ⋆, △, ⊙, ▽) represent the numerical results for a chain with 1000 particles at $t=10/\omega$, $50/\omega$, $100/\omega$, $500/\omega$, and $800/\omega$, respectively.

The wavelength at time t is

$$(x_f - x_b)\lambda(t) = 3.4826(\omega t)^{1/3}. \quad (34)$$

The changes of the wavelength with time is reflected in Fig. 9. We can find that the analytical solution can well reflect the changes of the wavelength.

VII. OTHER PROPERTIES FOR THE GRANULAR CHAIN

In HSJ theory, they qualitatively present that the energy involved in the impulse wave is equipartition between potential and kinetic forms, which is termed in Ref. [13] as the case for a harmonic potential. In the following, we will prove such property truly exists in the chain based on our analytical form.

The potential energy between two adjacent particles, n and $(n+1)$, can be expressed as

$$\frac{1}{2}[u(n+1,t)-u(n,t)]^2 = \frac{1}{2}\left(\int a(t)g_{n+1}dt - \int a(t)g_n dt\right)^2 = \frac{1}{2}\left(\int a(t)(g_{n+1}-g_n)dt\right)^2, \quad (35)$$

where, g_{n+1} and g_n represent the velocity of the particle n and $(n+1)$, which follows the form of Eq. (27). Retention of the first term in a Taylor series expansion of g_{n+1} about g_n leads to

$$\frac{1}{2}[u(n+1,t)-u(n,t)]^2 = \frac{1}{2}\left(\int a^3(t)g_n^{(1)}dt\right)^2. \quad (36)$$

Since

$$\frac{dg_n}{dt} = g_n^{(1)}\frac{d[a^2(t)(x - x_{\max} - \omega t)]}{dt} = g_n^{(1)}[-\omega a^2 + 2aa'(x - x_{\max})].$$

Treat $a(t)$ as constant because it changes slowly with time, we can get

$$\frac{dg_n}{dt} = -\omega a^2 g_n^{(1)}.$$

Then, Eq. (36) becomes

$$\frac{1}{2}[u(n+1,t)-u(n,t)]^2 = \frac{1}{2}\left(-\int \frac{a(t)}{\omega}dg_n\right)^2 = \frac{1}{2}\left(-\frac{a(t)}{\omega}g_n\right)^2 = \frac{1}{2\omega^2}[a(t)g_n]^2. \quad (37)$$

The ratio of the potential and kinetic energy involved in the impulse wave is

$$\frac{\frac{1}{2}m\sum[a(t)g_n]^2}{\frac{1}{2}k\sum[u(n+1,t)-u(n,t)]^2} = \frac{m\omega^2}{k} = 1. \quad (38)$$

Obviously, the kinetic energy equals the potential energy in the impulse wave. The equipartition of the energy in impulse wave is the typical property for the chain with linear contact force, which is also the factor resulting in diffusion of the wave. For a chain with Hertz contact force, the part of the potential form is greater than the one of kinetic form in the impulse wave. The extent of the wave diffusion must be lower than the chain with linear contact force.

Although the total momentum of the system is conservation, the forward momentum increases due to the particles ejected backward or lost contact as the pulse moves along. In terms of the function of the impulse wave, the forward momentum per unity mass is

$$K_f = \frac{\sum mV(n,t)}{m} = a(t)\lambda(t)\int_{x_b}^{x_f} g(x)dx = 0.9575(\omega t)^{1/6}. \quad (39)$$

Figure 10 shows our analytical result and the numerical simulation for a chain with 1000 particles. We can also find that our analytical result coincide well with the numerical simulation.

The changing rate of the forward momentum just equals the rebounding velocity of the last particle that loses contact from the chain. The impulse velocity is ω, the time of the particle n is ejected from the chain at $t=n/\omega$. The velocity of the ejected particle $v(n)$ can be expressed as

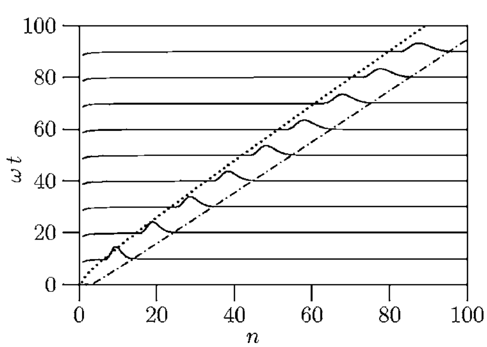

FIG. 9. The real line is numerical results of the wave shape for a chain in different time $t=10/\omega$, $20/\omega$, etc. The dotted curve is the position of the rear of the wave, which is expressed by the function, $n=2.6835+\omega t-2.8906(\omega t)^{1/3}$. The dashed-dotted line is the position of the front of the wave, which is expressed by the function $n=2.6835+\omega t+0.5920(\omega t)^{1/3}$.

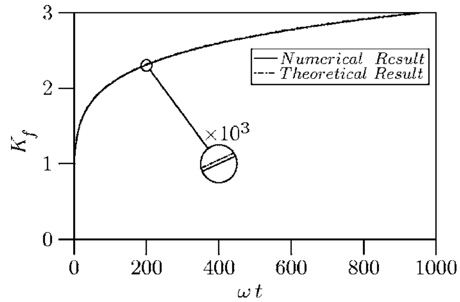

FIG. 10. The real line is the numerical simulation of the momentum that moves forward in a chain with 1000 particles as time goes by. The dashed-dotted line is our analytical result $K_f = 0.9575(\omega t)^{1/6}$ in Eq. (39).

$$v(n) = -\left.\frac{dK_f}{dt}\right|_{t=n/\omega} = -0.1596n^{-5/6}. \tag{40}$$

There is a discrepancy between our analytical result and the numerical simulation in HSJ for which the coefficient is 0.158. But the error is acceptable if the chain is long enough. Figure 11 shows the ratio of the rebounding velocity $\eta(n)$ based on our numerical results to the one in our analytical form expressed in Eq. (40). The maximal relative error appears at the first particle and is only about 18%.

If the chain is long but finite, containing N particles, some particles at the end of the chain will fly off with positive velocity when the wave arrives. Since the time that the wave reaches the end of the chain is about $t = N/\omega$, the wavelength at the end of the line is about $3.4826N^{1/3}$. The number of the particles flying off the line can be approximated as the half-wavelength at this moment, i.e., $1.7413N^{1/3}$ particles will fly off the chain. Meanwhile, the flying velocities of the particles should double the values of the posterior half of the speed wave at this moment, since the reflection of the wave is emitted from the free end,

$$v(n) = 2V(n, N/\omega) = 2.3404N^{-1/6} g\left(\frac{n - N}{0.7302N^{1/3}} + 5.2515\right). \tag{41}$$

The fastest in the flying particles is at the end of the line with a velocity $v(N) = 1.386N^{-1/6}$.

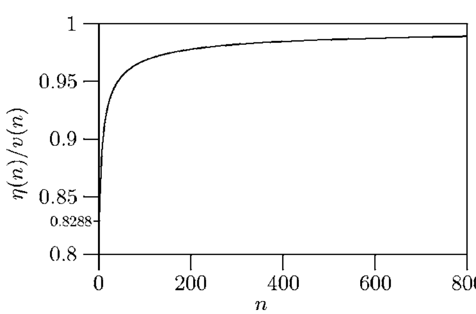

FIG. 11. $\eta(n)$ is a numerical result of the speed after impact for the nth particle in a chain with 1000 particles. $v(n)$ is our analytical result in Eq. (40). The real line is the ratio $\eta(n)/v(n)$, and it approaches 1 with the increasing of n.

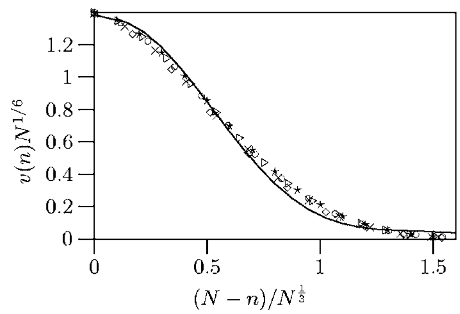

FIG. 12. The flying off velocity of the ball are shown in this figure. The real line is our analytical solution in Eq. (41). The numerical results for the chains, with N=200, 400, 600, 800, 1000, represented by ◇, ×, ∘, ▷, ⋆, respectively.

In HSJ theory, there are $1.5N^{1/3}$ particles that have a significant velocity and the fastest at the end with a velocity $v(N) = 1.397N^{-1/6}$ based on their numerical results. Figure 12 compares our analytical solution with numerical results, and we can also find that the analytical solution can well reflect the velocity distribution of the particles flying off the chain.

VIII. CONCLUSION

In this paper, we have studied the impulse dynamics for a 1D granular chain where the contact force is linear. Our interest has been in establishing a theoretical model for such chain when it is hit by another equal particle.

First we obtained theoretical solutions for a shorter chain with different numbers of particles. We find that a correct shape of impulse wave can be initially generated if the number of the particles in the chain is greater than 6. In terms of the properties of theoretical solutions and using least square method, we have developed a continuous function to represent the initial impulse wave. Furthermore, we have constructed a wave equation of continuum approximation for 1D granular chain, which agrees well with the HSJ theory.

Then we follow the assumptions presented by HSJ that the amplitude of the wave decreases and its wavelength increases during traveling. The initial wave profile obtained from the theoretical results of a chain with seven particles is modified by two functions $a(t)$ and $\lambda(t)$. We have obtained the analytical solution for describing the impulse wave propagation. The impulse wave propagates forward with the speed ω, its width grows at $0.7302(\omega t)^{1/3}$, and the amplitude decreases at $1.1702(\omega t)^{-1/6}$.

We also studied other properties of the impulse wave during traveling. Based on the analytical model, we have proved that the energy involved in the impulse wave is equipartition between potential and kinetic forms. The forward momentum per unity mass is $0.9575(\omega t)^{1/6}$, which increases with time due to the rebounding motion of the particles. The speed of the ejected granules decreases, and the nth granule being ejected with a speed $-0.1596n^{-5/6}$. For a long but finite chain, there are $1.7413N^{1/3}$ particles at the far end flying off the chain and the last particle have the fastest velocity $v(N)=1.386N^{-1/6}$. We supported our analytical solutions via numerical simulation.

A possible extension of this work is consideration of the effect of the mass and stiffness of the impacting particle. Work along these directions is in progress.

ACKNOWLEDGMENTS

The support of the National Science Foundation of China (Grants Nos. 10272002; 60334030) and the Foundation of Engineering Research Institute of Peking University (204035) is gratefully acknowledged.

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